Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

He proceeds with the demonstrations, occupying two and one half pages of the *Quarterly*, avoiding the methods he criticizes and that by differentiation referred to in Mr. Finkel's problem, closing with the words "the results here obtained coincide, it may be remarked, with those given in Todhunter's *Int. Cal.*"

To solve Professor Finkel's problem we need only put $\alpha = 0$ in the first of the integrals mentioned by Walton.

437 (Calculus). Proposed by LEIGH PAGE, Yale University.

Integrate

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,dx$$

without the use of the gamma functions.

I. Solution by Oscar S. Adams, Coast and Geodetic Survey, Washington, D. C.

By direct integration, we have

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}.$$

Let $a = \alpha - i\beta$. Then

$$\int_0^\infty e^{-\alpha x + i\beta x}dx = \frac{1}{\alpha - i\beta} = \frac{\alpha + i\beta}{\alpha^2 + \beta^2},$$

or

$$\int_0^\infty e^{-\alpha x}(\cos \beta x + i \sin \beta x)dx = \frac{\alpha + i\beta}{\alpha^2 + \beta^2}.$$

By equating real and imaginary parts we obtain the two definite integrals

$$\int_0^\infty e^{-\alpha x} \cos \beta x dx = \frac{\alpha}{\alpha^2 + \beta^2}, \quad \text{and} \quad \int_0^\infty e^{-\alpha x} \sin \beta x dx = \frac{\beta}{\alpha^2 + \beta^2}.$$

Since the first of these integrals is a uniform function of β, we have the relation

$$\int_0^\beta d\beta \int_0^\infty e^{-\alpha x} \cos \beta x dx = \int_0^\infty dx \int_0^\beta e^{-\alpha x} \cos \beta x d\beta = \int_0^\beta \frac{\alpha\, d\beta}{\alpha^2 + \beta^2};$$

or

$$\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x}\,dx = \tan^{-1}\frac{\beta}{\alpha}.$$

This integral is also a uniform function of β in the region $+\frac{\pi}{2} \geqq \tan^{-1}\frac{\beta}{\alpha} \geqq -\frac{\pi}{2}$.

Hence, in this domain, we have

$$\int_0^\infty dx \int_0^2 e^{-\alpha x} \frac{\sin \beta x}{x}\,d\beta = \int_0^2 \tan^{-1}\frac{\beta}{\alpha}\,d\beta$$

or

$$\int_0^\infty \frac{e^{-\alpha x}(1 - \cos 2x)}{x^2}\,dx = 2 \tan^{-1}\frac{2}{\alpha} - \frac{\alpha}{2}\log(\alpha^2 + 4).$$

This is a uniform function of α. Hence, α can converge to zero. This gives

$$\int_0^\infty \frac{2 \sin^2 x}{x^2}\,dx = 2 \cdot \frac{\pi}{2} = \pi, \quad \text{or} \quad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \tfrac{1}{2}\pi.$$

Since $\sin^2 x/x^2$ is an even function of x, we have

$$\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\,dx = 2\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \pi.$$

II. Solution by G. Paaswell, N. Y. City.

Since $\sin^2 x/x^2$ is an even function of x we have

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = 2 \int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$

Place

$$\int_0^{\infty} \frac{\sin^2 ax}{x^2} dx = u; \quad \text{then} \quad \frac{du}{da} = \int_0^{\infty} \frac{\sin 2ax}{x} dx = \frac{\pi}{2}.$$

Hence, $u = \pi a/2$. Upon placing $a = 1$, we have

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}, \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

III. Solution by the Proposer.

The definite integral

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2} dx$$

is one of frequent occurrence in theoretical physics. As the text-books invariably seem to use gamma functions in evaluating this integral, the following integration without the use of other than the elementary functions may be of interest to lecturers in theoretical physics and to their students. We have

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2} dx = 2 \int_0^{\infty} \frac{\sin^2 x}{x^2} dx$$

since the integrand is an even function of x.

Consider

$$\int_0^{\infty} \int_0^{\infty} \int_y^{\infty} e^{-yx} \sin^2 x dx dy dy = \int_0^{\infty} \int_0^{\infty} e^{-yx} \frac{\sin^2 x dx dy}{x} = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$

Since the limits for the y integration are not functions of x,

$$\int_0^{\infty} \int_0^{\infty} \int_y^{\infty} e^{-yx} \sin^2 x dx dy dy = \int_0^{\infty} \int_y^{\infty} \int_0^{\infty} e^{-yx} \sin^2 x dy dy dx$$
$$= \frac{1}{2} \int_0^{\infty} \int_y^{\infty} dy dy \left[\int_0^{\infty} e^{-yx} dx - \int_0^{\infty} e^{-yx} \cos 2x dx \right].$$

Now

$$\int_0^{\infty} e^{-yx} dx = \frac{1}{y} \quad \text{and} \quad \int_0^{\infty} e^{-yx} \cos 2x dx = \frac{y}{y^2 + 4}.$$

Hence,

$$\int_0^{\infty} \int_0^{\infty} \int_y^{\infty} e^{-yx} \sin^2 x dx dy dy = 2 \int_0^{\infty} \int_y^{\infty} \frac{dy dy}{y(y^2 + 4)} = \frac{1}{4} \int_0^{\infty} \log\left(1 + \frac{4}{y^2}\right) dy.$$

Integrating the last expression by parts,

$$\frac{1}{4} \int_0^{\infty} \log\left(1 + \frac{4}{y^2}\right) dy = \frac{1}{4}\left[y \log\left(1 + \frac{4}{y^2}\right)\right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} \frac{dy}{1 + \frac{y^2}{4}} = \frac{1}{4}\left[y \log\left(1 + \frac{4}{y^2}\right)\right]_0^{\infty} + \frac{\pi}{2}.$$

The expression $y \log [1 + (4/y^2)]$ is indeterminate for both limits. Evaluating it by the usual methods, it becomes zero for each. Hence,

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

Solutions similar to the second above were received from William Hoover and Paul Capron.

Solutions were received too late for credit in a previous issue of the Monthly as follows: From Paul Capron, 490 and 491 Algebra and 522 Geometry; from C. P. Sousley, 486 and 488 Algebra and 518, 520, 521, and 522 Geometry; from William Hoover, 521 and 522 Geometry; and from C. C. Yen, 490 Algebra and 522 Geometry.